Controllability
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Problem 1 The linearized controlled dynamics of the spatial restricted 3-body problem in the vicinity of a colinear equilibrium point $L_i,\ i = 1, 2, 3,$ can be written in the form

$$\dot{v} = Av + Bu$$

with

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_2 \\ 0 & 0 & 0 & -\lambda_1 & 0 & 0 \\ 0 & -\omega_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_2 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} \frac{2\lambda_1}{s_1} & -\frac{\lambda_1^2 - 2c2 - 1}{s1} & 0 \\ -\frac{2\omega_1}{s_2} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2\lambda_1}{s_1} & \frac{\lambda_1^2 - 2c2 - 1}{s1} & 0 \\ 0 & \frac{-\omega_1^2 - 2c2 - 1}{s2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\omega_2}} \end{pmatrix}$$

where $\lambda_1,\ \omega_1 \neq \omega_2,\ c_2,\ s_1,\ s_2$ are positive constants. Assume that there is no constraint on the control bound. Show that this system is controllable.

Problem 2 Show the Hautus Lemma: let A be a $n \times n$ matrix and B be a $n \times m$ matrix, where n and m are 2 positive integers. The following conditions are equivalent

(1) the pair (A, B) satisfies the Kalman condition

$$\text{rank}(B, AB, \ldots, A^{n-1}B) = n$$

(2) $\forall\ \lambda \in \mathbb{C},\ \text{rank}(\lambda I - A, B) = n$

(3) $\forall\ \lambda \in \text{Spec}(A),\ \text{rank}(\lambda I - A, B) = n$

(4) $\forall z$ eigenvector of A^T, $B^T z \neq 0$

(5) $\exists c > 0 \mid \forall \lambda \in \mathbb{C}, \forall z \in \mathbb{R}^n, ||(\lambda I - A^T)z||^2 + ||B^T z||^2 \geq c||z||^2$.

Problem 3 We consider the controlled Euler equation

$$\begin{pmatrix} \dot{\omega_1}(t) \\ \dot{\omega_2}(t) \\ \dot{\omega_3}(t) \end{pmatrix} = \begin{pmatrix} \omega_2(t)\omega_3(t) + u(t)b_1 \\ -\omega_1(t)\omega_3(t) + u(t)b_2 \\ \omega_1(t)\omega_2(t) + u(t)b_3 \end{pmatrix}$$

where $b = (b_1, b_2, b_3)$ is constant and $||u(t)|| \leq 1$.

1. Assume that $u = 0$. Show that the differential equation is Poisson-stable.
2. Give a necessary and sufficient condition on b for the system to be controllable.